Finite Element Solver of a Poisson Equation in One Dimension

We have an equation:

$$-\frac{d^2u}{dx^2} = g(x), \quad 0 < x < 1$$

With g(x) defined as:

$$g(x) = 2\sin(\pi x) + 4\pi(x-1)\cos(\pi x) - \pi^2(x-1)^2\sin(\pi x)$$

With Dirichlet boundary condition:

$$u(x=0) = 0, \quad \frac{du(x=1)}{dx} = 0$$

Assignment 1

We have the function:

$$u(x) = -(x-1)^2\sin(\pi x).$$

Then:

$$\frac{du(x)}{dx} = -2\sin(\pi x)(x-1) - (x-1)^2\pi\cos(\pi x)$$

$$\frac{d}{dx}\left(\frac{du(x)}{dx}\right) = -2\sin(\pi x) - 2\pi\cos(\pi x)(x-1) - 2(x-1)\pi\cos(\pi x) + (x-1)^2\pi^2\sin(\pi x)$$
$$= -2\sin(\pi x) - 4\pi(x-1)\cos(\pi x) + \pi^2(x-1)^2\sin(\pi x) = -g(x)$$

So:

$$-\frac{d^2u}{dx^2} = g(x)$$

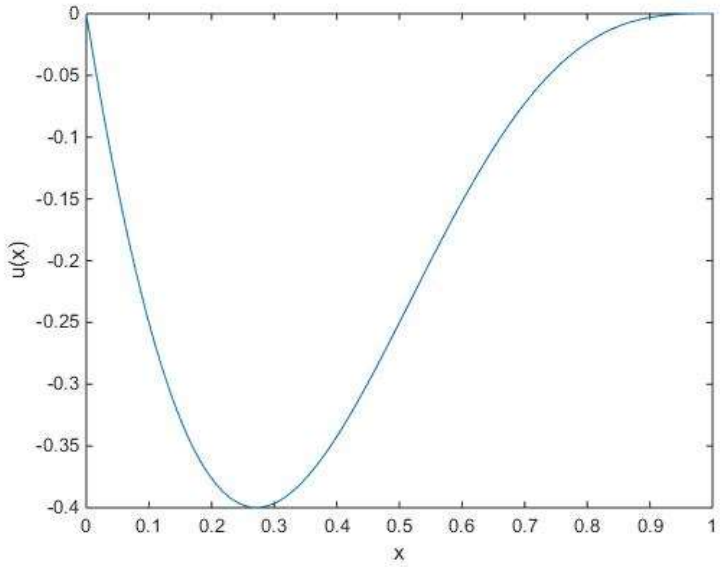
Also:

$$u(0) = -(0-1)^2\sin(\pi 0) = -(-1)^2\sin(0) = 0, \frac{du(x=1)}{dx} = -2\sin(\pi)(1-1) - (1-1)^2\pi\cos(\pi) = 0$$

We wrote the next matlab function to plot this analytical solution:

```
clear all;
clc;
% -- function u(x)
x=0:0.001:1;
ux=-(x-1).^2.*sin(pi*x);
figure(1)
plot(x,ux)
xlabel('x')
ylabel('u(x)')
```

And here is the figure for u(x)



Assignment 2.

Here, we can write the numerical scheme to get the algorithm for numerical solution of our equation.

We know, that the second derivative at some point i can be represented as next:

$$\frac{du^2}{d^2x} = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$$

Thus, we can write en equation of form:

$$\frac{(-1;\ 2; -1)}{2\Delta x} * \begin{pmatrix} u_{i+1} \\ u_i \\ u_{i-1} \end{pmatrix} = g_i(\Delta x)$$

So, as we see we can write the next equation:

$$Au =$$

Where:

- A is tridiagonal matrix (non-zero elements only on main diagonal and on elements up and beneath from it).
- u is unknown vector of function we looking at.
- g is a vector with function g(x) values at points x_i.

Assignment 3.

The trapezoidal rule can be written is the next way :

$$\int_a^b f(x)dx \quad \frac{b-a}{2}\big(f(b) + f(a)\big)$$

We can write the our intermediate matrix S_{ei} which appear for every step and every element e_i as the next (here we use the result for matrix A from previous part):

$$S_{ei} = \frac{1}{e_i}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Using previous part and trapezoidal rule we can write the vector f_{ei} as the next:

$$f_{ei} = \frac{e_i}{2}\begin{pmatrix} g_i \\ g_{i+1} \end{pmatrix}$$

Assignment 4.

The next step will be to assemble all our element matrixes S_{ei} in one global matrices, for example if we only have 2 nodes in our system we can assemble matrices S_{e1} and S_{e2} in the next way:

$$S_{e1} = \frac{1}{e_1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, S_{e2} = \frac{1}{e_2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \; S = [S_{e1}, S_{e2}] = \begin{pmatrix} 1/e_1 & -1/e_1 & 0 \\ -1/e_1 & 1/e_1 + 1/e_2 & -1/e_2 \\ 0 & -1/e_2 & 1/e_2 \end{pmatrix}$$

So, we see that for N=2 nodes we have 3x3 matrix , this will work also in the general case when we have N nodes and (N+1)x(N+1) global matrix.

The same induction we can build for vector f_{ei}, for example for 2 nodes we will have:

$$f_{e1} = \frac{e_1}{2}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad f_{e2} = \frac{e_2}{2}\begin{pmatrix} g_2 \\ g_3 \end{pmatrix}, \quad f = \begin{pmatrix} \frac{e_1 g_1}{2} \\ e_2 g_2 \\ \frac{e_3 g_3}{2} \end{pmatrix}$$

Assignment 5.

This was done as additional function in Matlab:

```
function f=GenerateMesh(xo,xe,N)
%  --- create the mesh of N+1 point in range x0 ... xe
dx=(xe-xo)/N;
for i=1:N+1
   x(i)=xo+dx*(i-1);
end
f=x;
end
```

Assignment 6.

This was done as additional function in Matlab:

```
function f=SourceFct(xi)
%  -- gives value of source function at xi
f=2*sin(pi*xi)+4*pi*(xi-1)*cos(pi*xi)-pi^2*(xi-1)^2*sin(pi*xi);
end
```

Assignment 7.

This was done as additional function in Matlab:

```
function f=GenerateTopology(N)
%  -- generate topology matrix
f=zeros(N,2);
for i=1:N
   f(i,1)=i;
   f(i,2)=i+1;
end
end
```

Assignment 8.

This was done as additional function in Matlab:

```
function f=GenerateElementMatrix(xi,xi1)
%  -- input , the values of x(i) and x(i+1)
Sei=[1 -1; -1 1];
Sei=Sei/(xi1-xi);
f=Sei;
end
```

Assignment 9.

This was done as additional function in Matlab:

```
function f=GenerateElementVector(xi,xi1)
gi=SourceFct(xi);
gi1=SourceFct(xi1);
f=[gi*(xi1-xi)/2;gi1*(xi1-xi)/2];
end
```

Assignment 10.

This was done as additional function in Matlab:

```
function f=AssembleMatrix(xo,xe,N)
S=zeros(N+1,N+1);
x=GenerateMesh(xo,xe,N);
elmat=GenerateTopology(N);
for i=1:N
        Sei=GenerateElementMatrix(x(i),x(i+1));
   for j=1:2
      for k=1:2
      S(elmat(i,j),elmat(i,k))=S(elmat(i,j),elmat(i,k))+Sei(j,k);
      end
   end
end
f=S;
end
```

Assignment 11.

This was done as additional function in Matlab:

```
function f=AssembleVector(xo,xe,N)
 fv=zeros(N+1,1);
x=GenerateMesh(xo,xe,N);
elmat=GenerateTopology(N);
for i=1:N
    fei=GenerateElementVector(x(i),x(i+1));
```

```
  for j=1:2
    fv(elmat(i,j))=fv(elmat(i,j))+fei(j);
  end
end
f=fv;
end
```

Assignment 12.

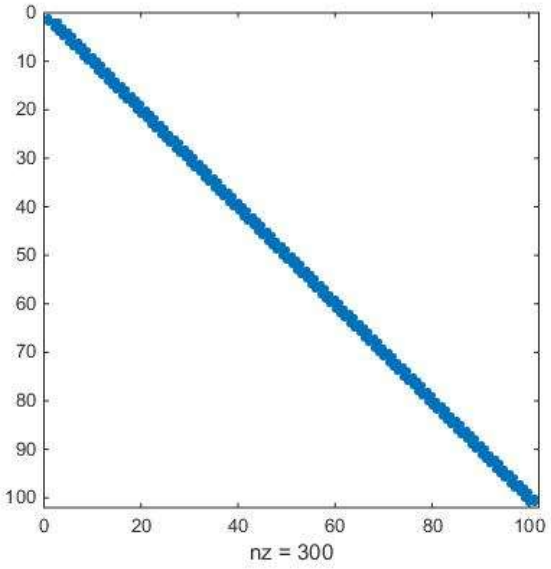
This was reached by adding next lines to main program when we calculate the matrix S and vector f before use them to solve en equation :

$$Su^h = f$$

```
% boundary ---
S(1,1)=1;
S(1,2)=0;
f(1)=0;
```

Assignment 13.

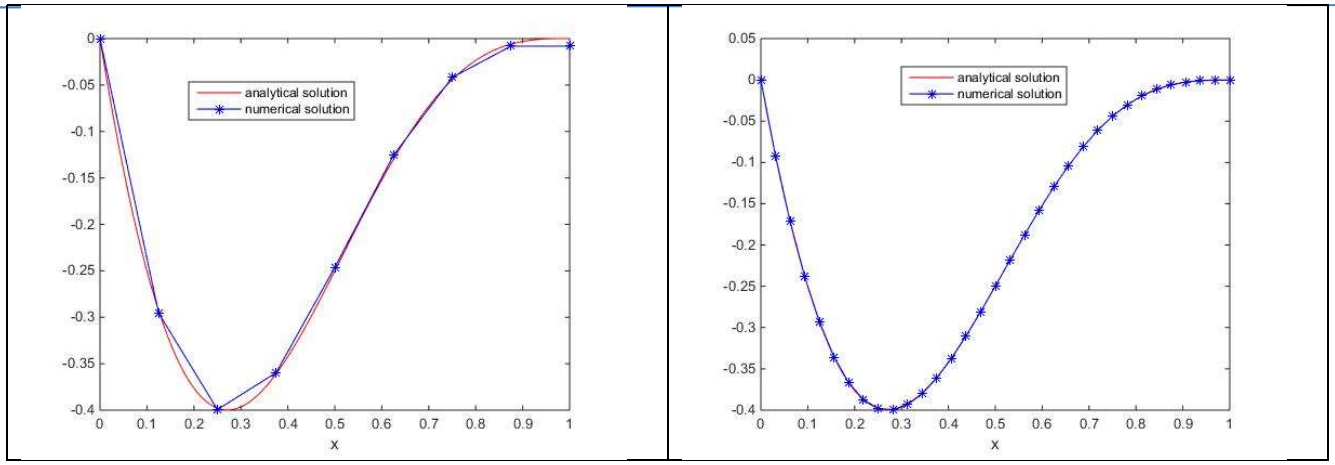
We run our main program for N=100 and got the next visualization for S matrix as



Assignment 14.

Here is the graphs where we have two solutions, analytical u(x) from first part and numerical for N=8, and N=32.

N=8	N=32



As we see – for more number of nodes we have better convergence between two solutions, analytical and numerical.

Elective Part

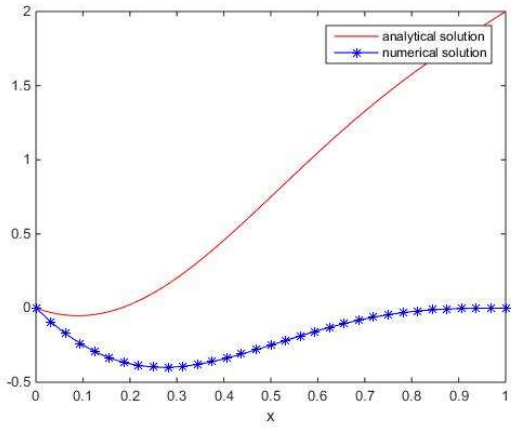
Part 1

Lets try to write analytical solution as:

$$u(x) = -(x-1)^2 \sin(\pi x) + 2x$$

This will give the same result for second derivative as we know that $(2x)'' = 0$

So, we after changes in main and additional function we will get the result.

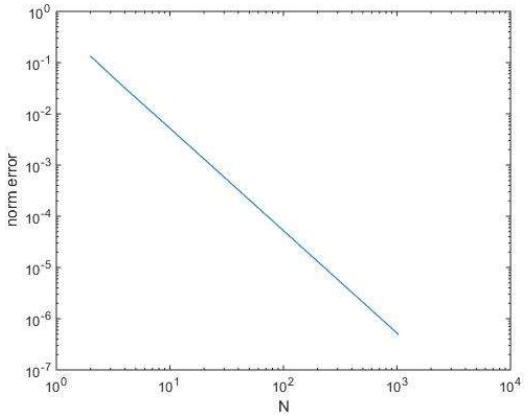


As we see the results very unclose – this can be explained that we need change not only analytical solution but also the boundary conditions.

Part 2

To compare results for different number of nodes we can write the loop for different N=2,4,8,16,… etc. On every loop we calculate the numerical solution and the error (largest values for mesh).

And we got the next graph for norm of error vs. number of nodes (see q_err.m file):



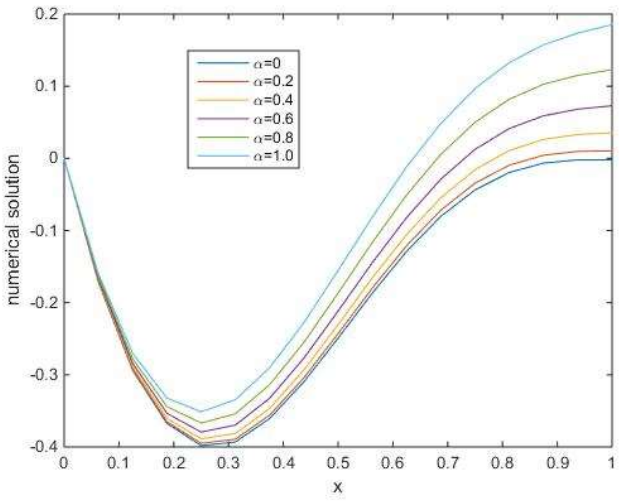
As we see , in logarithmic axes we have linear dependence for logarithm of norm of error from logarithm from number of nodes, this leads from the way we wrote our equations using linear (first order) Lagrangian shape functions.

Part 3

Here we changed the program by adding to last element in vector f the value:

$$f(last) = f(last) + \alpha h, \quad h = 1/N$$

So, in this way we got the next graph with different numerical solutions (for N=16) for different values of α (see q_neumann.m for script):



As we see – with increasing of α we get the increasing of left boundary of function