Tutorial 1

1. Show that the differential dG_p of a function G, from a surface S to $\mathbb{R}^3$ is linear.

(proof) Let be $G, F \in C^\infty(S)$. It is sufficient that for any $\lambda, \mu \in \mathbb{R}$,

$$d(\lambda F + \mu G) = \lambda dF + \mu dG.$$

Assume that $p \in S,\ X_p \in T_p S$. Then

$$d(\lambda F + \mu G)_p X_p = X_p(\lambda F + \mu G) = \lambda X_p F + \mu X_p G =$$
$$= \lambda dF_p X_p + \mu dG_p X_p = (\lambda dF + \mu dG) X_p$$

(end)

2. Calculate the Gauss map, the Wiengarten map and the principal curvatures for
(a) A sphere of radius, R,

(solution) $x = R\cos u \cos v,\ y = R\cos u \sin v,\ z = R\sin u$!

$$\dot{x}_u = -R\sin u \cos v,\ \dot{y}_u = -R\sin u \sin v,\ \dot{z}_u = R\cos u$$
$$\dot{x}_v = -R\cos u \sin v,\ \dot{y}_v = -R\cos u \cos v,\ \dot{z}_v = 0$$

$$E = R^2, F = 0, G = R^2\cos^2 u$$

$$H = \sqrt{EG - F^2} = R^2 \cos u$$

$$\overrightarrow{r_{u^2}} = \left(-R\cos u \cos v, -R\cos u \sin v, -R\sin u\right)$$
$$\overrightarrow{r_{uv}} = \left(R\sin u \sin v, -R\sin u \cos v, 0\right)$$
$$\overrightarrow{r_{v^2}} = \left(-R\cos u \cos v, -R\cos u \sin v, 0\right)$$

$$\therefore (\overrightarrow{r_u}, \overrightarrow{r_v}, \overrightarrow{r_{u^2}}) = R^3 \cos u,\ \ (\overrightarrow{r_u}, \overrightarrow{r_v}, \overrightarrow{r_{uv}}) = 0,\ \ (\overrightarrow{r_u}, \overrightarrow{r_v}, \overrightarrow{r_{v^2}}) = R^3\cos^3 u$$

$$L = -\frac{1}{H}(\overrightarrow{r_u}, \overrightarrow{r_v}, \overrightarrow{r_{u^2}}) = -\frac{R^3\cos u}{R^2 \cos u} = -R$$
$$M = -\frac{1}{H}(\overrightarrow{r_u}, \overrightarrow{r_v}, \overrightarrow{r_{uv}}) = 0$$
$$N = -\frac{1}{H}(\overrightarrow{r_u}, \overrightarrow{r_v}, \overrightarrow{r_{v^2}}) = -\frac{R^3\cos^3 u}{R^2\cos u} = -R\cos^2 u$$

-weingarten map:

$$\mathrm{A} = \frac{1}{EG - F^2}\begin{pmatrix} GL - FM & GM - FN \\ -FL + EM & -FM + EN \end{pmatrix} = \frac{1}{R^4\cos^2 u}\begin{pmatrix} -R^3\cos^2 u & 0 \\ 0 & -R^3\cos^2 u \end{pmatrix} = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{pmatrix}$$

-Gauss map:

$$\vec{r_u} \times \vec{r_v} = -R^2(\cos^2 u \cos v, \cos^2 u \sin v, \sin u \cos v),$$

$$\vec{n} = \frac{\vec{r_u} \times \vec{r_v}}{\left\| \vec{r_u} \times \vec{r_u} \right\|} = \frac{-R^2}{R^2 \cos u}(\cos^2 u \cos v, \cos^2 u \sin v, \sin u \cos v)$$

-principle curvature:

$$k_1 = k_2 = -\frac{1}{R}.$$

(b) A surface of revolution given by the curve $x = f(z)$ rotated about the z axis, and (solution)

$$\begin{cases} x = f(z)\cos\varphi \\ y = f(z)\sin\varphi \\ z = z \end{cases}$$

$$\vec{r} = (f(z)\cos\varphi, f(z)\sin\varphi, z)$$

$$\vec{r_z}' = (f'(z)\cos\varphi, f'(z)\sin\varphi, 1)$$

$$\vec{r_\varphi}' = (-f(z)\sin\varphi, f(z)\cos\varphi, 0)$$

$$E = \left(\vec{r_z}'\right)^2 = f'(z)^2 + 1,\ F = 0,\ G = \left(\vec{r_\varphi}'\right)^2 = f(z)^2,\ H = \sqrt{EG - F^2} = f(z)\sqrt{f'(z)^2 + 1}$$

$$\vec{r_{z^2}} = (f''(z)\cos\varphi, f'(z)\sin\varphi, 0)$$

$$\vec{r_{z\varphi}} = (-f'(z)\sin\varphi, f'(z)\cos\varphi, 0)$$

$$\vec{r_{\varphi^2}} = (-f(z)\cos\varphi, -f(z)\sin\varphi, 0)$$

$$(\vec{r_z}, \vec{r_\varphi}, \vec{r_{z^2}}) = -f(z)f''(z),\quad (\vec{r_z}, \vec{r_\varphi}, \vec{r_{z\varphi}}) = 0,\quad (\vec{r_z}, \vec{r_\varphi}, \vec{r_{\varphi^2}}) = f^2(z)$$

$$\therefore L = -\frac{f(z)f''(z)}{f(z)\sqrt{f'(z)^2 + 1}} = -\frac{f''(z)}{\sqrt{f'(z)^2 + 1}},$$

$$M = -\frac{0}{f(z)\sqrt{f'(z)^2 + 1}} = 0,\ N = -\frac{f(z)^2}{f(z)\sqrt{f'(z)^2 + 1}} = -\frac{f(z)}{\sqrt{f'(z)^2 + 1}}$$

-weingarten map:

$$\mathrm{A} = \frac{1}{f^2(z)(f'(z)^2 + 1)} \begin{pmatrix} -f^2(z) \cdot \frac{f''(z)}{\sqrt{f'(z)^2 + 1}} & 0 \\ 0 & -(f'(z)^2 + 1) \cdot \frac{f(z)}{\sqrt{f'(z)^2 + 1}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{f''(z)}{(f'(z)^2 + 1)^{3/2}} & 0 \\ 0 & -(f'(z)^2 + 1)^{1/2} \cdot f(z) \end{pmatrix}$$

-Gauss map:

$$\vec{n} = \frac{1}{f(z)\cdot\sqrt{f'(z)^2+1}}\left(-f(z)\cos\varphi, -f(z)\sin\varphi, f(z)f'9z)\right)$$

-principle curvature:

$$k_1 = -\frac{f''(z)}{(f'(z)^2+1)^{3/2}},\ k_2 = -f(z)\cdot\sqrt{f'(z)+1}$$

(c) The surface of revolution about the $z-$axis of a circle in the $xz-$plane with center $(d,0,0)$ with radius $r<d$.

(solution)

$$\begin{cases} x = (d + r\cos u)\cos v \\ y = (d + r\cos u)\sin v \\ z = r\sin u \end{cases}$$

$$\vec{r_u} = \left(-r\sin u\cos v, -r\sin u\sin v, r\cos u\right)$$

$$\vec{r_v} = \left(-(d+r\cos u)\sin v, (d+r\cos u)\cos v, 0\right)$$

$$E = \vec{r_u}^2 = r^2,\ F = 0,\ G = \vec{r_v}^2 = (d+r\cos u)^2,\ H = \sqrt{EG-F^2} = r(d+r\cos u)$$

$$\vec{r_u}\times\vec{r_v} = \left(-r(d+r\cos u)\cos u\cos v, -r(d+r\cos u)\cos u\sin v, -r(d+r\cos u)\sin u\right)$$

$$\overrightarrow{r_{u^2}} = (-r\cos u\cos v, -r\cos u\sin v, -r\sin u)$$

$$\overrightarrow{r_{uv}} = (r\sin u\sin v, -r\sin u\cos v, 0)$$

$$\overrightarrow{r_{v^2}} = (-(d+r\cos u)\cos v, -(d+r\cos u)\sin v, 0)$$

$$L = -\frac{r^2(d+r\cos u)}{r(d+r\cos u)} = -r,\ M = 0,\ N = -\frac{r(d+r\cos u)\cos u}{r(d+r\cos u)} = -\cos u$$

-weingarten map:

$$A = \frac{1}{r^2(d+r\cos u)^2}\begin{pmatrix} (d+r\cos u)^2(-r) & 0 \\ 0 & -r^2\cos u \end{pmatrix} = \begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{\cos u}{(d+r\cos u)^2} \end{pmatrix}$$

-Gauss map:

$$\vec{n} = -(\cos u\sin v, \cos u\sin v, \sin u)$$

-principle curvature:

$$k_1 = \frac{1}{r}, k_2 = \frac{\cos u}{(d+r\cos u)^2}$$

(d) The surface parametrized by

$$r(u,v) = \left(u - u^3/3 + uv^2, v - v^3/3 + vu^2, u^2 - v^2\right).$$

(solution)

$$\vec{r_u} = (1-u^2+v^2, 2uv, 2u)$$
$$\vec{r_v} = (2uv, 1-v^2+u^2, -2v)$$
$$E = \vec{r_u}^2 = (1+u^2+v^2)^2,\ F=0,\ \ G=(1+u^2+v^2)^2,\ H=(1+u^2+v^2)^2$$
$$\vec{r_{u^2}} = (-2u, 2v, 2),\ \ \vec{r_{uv}} = (2v, 2u, 0),\ \ \vec{r_{v^2}} = (2u, -2v, -2)$$
$$\vec{r_u}\times\vec{r_v} = \left(-2u(u^2+v^2+1), 2v(u^2+v^2+1), 1-(u^2+v^2)^2\right)$$
$$L = -\frac{4(1+u^2+v^2)}{(1+u^2+v^2)^2} = -\frac{4}{1+u^2+v^2},\ M=0,$$
$$N = -\frac{-4(1+u^2+v^2)}{(1+u^2+v^2)^2} = \frac{4}{1+u^2+v^2}$$

-weingarten map:

$$A = \frac{1}{(1+u^2+v^2)^4}\begin{pmatrix} -4(1+u^2+v^2) & 0 \\ 0 & 4(1+u^2+v^2)\end{pmatrix} = \begin{pmatrix} -\frac{4}{(1+u^2+v^2)^3} & 0 \\ 0 & \frac{4}{(1+u^2+v^2)^3}\end{pmatrix}$$

-Gauss map:

$$\vec{n} = \frac{1}{(1+u^2+v^2)^2}(-2u(1+u^2+v^2), 2v(1+u^2+v^2), 1-(u^2+v^2))$$
$$= \left(-\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2}\right)$$

-principle curvature:

$$k_1 = -\frac{4}{(1+u^2+v^2)^3},\ \ k_2 = \frac{4}{(1+u^2+v^2)^3}$$

Tutorial 2

1. Show that the second fundamental form II_p is symmetric.

   (proof) we take two arbitrary tangent vectors $\xi, \eta \in T_pS$ and two arbitrary real number $\alpha, \beta \in$. Then we have, first of all:

$$\varphi_2(\eta,\xi) = -\varphi_1(A(\eta),\xi) \underset{\text{self-adjoint}}{\overset{A}{=}} -\varphi_1(\eta, A(\xi)) \underset{\text{symmetrical}}{\overset{\varphi_1}{=}} -\varphi_1(A(\xi),\eta) = \varphi_2(\xi,\eta),$$

which means that φ_2 is symmetrical. (end)

2. Show that the elementary symmetric functions $S_i(k_1,\cdots,k_{n-1})$ are the coefficient of x^i in the expansion of $(1+k_1x)\cdots(1+k_{n-1}x)$.
   (proof) When $n=2$,

$$(1+k_1x)(1+k_2x)=1+(k_1+k_2)x+k_1k_2x^2=1+S_1x+S_2x^2.$$

   Therefor $S_0=1,\ S_1(k_1,k_2)=k_1+k_2,\ S_2(k_1,k_2)=k_1k_2$.
   When $n=3$,

$$(1+k_1x)(1+k_2x)(1+k_3x)=1+(k_1+k_2+k_3)x+$$
$$+(k_1k_2+k_2k_3+k_1k_3)x^2+k_1k_2k_3x^3$$

So $S_0=1,\ S_1=k_1+k_2+k_3,\ S_2=k_1k_2+k_2k_3+k_1k_3,\ S_3=k_1k_2k_3$.

…

$$(1+k_1x)(1+k_2x)\cdots(1+k_nx)=1+x\sum_{i=1}^{n}k_i+x^2\sum_{i_1<i_2}k_{i_1}k_{i_2}+$$
$$+\cdots+x^nk_1k_2\cdots k_n$$

so $S_0=1,\ S_1=\sum_{i=1}^{n}k_i,\ \ S_2=\sum_{i_1<i_2}k_{i_1}k_{i_2},\ \cdots,\ S_{n-1}=k_1\cdots k_n$.

3. Calculate the frames for the sphere based on
   (a) the standard parameterization

   (solution)

$$\vec{r}=(\cos u\cos v,\ \cos u\sin v,\ \sin u)$$
$$\vec{r_u}=(-\sin u\cos v,\ -\sin u\sin v,\ \cos u)$$
$$\vec{r_v}=(-\cos u\sin v,\ \cos u\cos v,\ 0)$$

   (b) stereographic projection
4. Calculate frames for
   (a) The torus

   (solution) the equation

$$\left.\begin{aligned}x&=(a+b\cos u)\cos v\\ y&=(a+b\cos u)\sin v\\ z&=b\sin u\end{aligned}\right\}.$$

$$\overset{\circ}{\vec{r_u}}=(\dot{x}_u,\dot{y}_u,\dot{z}_u)=(-b\sin u\cos v,-b\sin u\sin v,b\cos u)$$
$$\overset{\circ}{\vec{r_v}}=(\dot{x}_v,\dot{y}_v,\dot{z}_v)=(-(a+b\cos u)\sin v,(a+b\cos u)\cos v,0)$$

   (b) The catenoid

$$x = a\cosh\frac{u}{a}\cos v,\ \ y = a\cosh\frac{u}{a}\sin v,\ \ z = u$$

$$\vec{r_u} = \left(\sinh\frac{u}{a}\cos v, \sinh\frac{u}{a}\sin v, 1\right)$$

$$\vec{r_v} = \left(-a\cosh\frac{u}{a}\sin v, a\cosh\frac{u}{a}\cos v, 0\right)$$

Tutorial 3

1. Calculate the first fundamental form for

(a) The sphere of radius, R,

(solution) $\vec{r} = (R\cos u\cos v, R\cos u\sin v, R\sin u)$,

$$\therefore \begin{aligned} \vec{r_u} &= \frac{\partial \vec{r}}{\partial u} = \left(-R\sin u\cos v, -R\sin u\sin v, R\cos u\right) \\ \vec{r_v} &= \frac{\partial \vec{r}}{\partial v} = \left(-R\cos u\sin v, -R\cos u\cos v, 0\right) \end{aligned}$$

$$g_{11} = \vec{r_u}^2 = R^2,\ \ g_{12} = \vec{r_u}\vec{r_v} = 0,\ \ g_{22} = \vec{r_v}^2 = R^2\cos^2 u\ .$$

Therefor the first fundamental form is the following:

$$R^2 du^2 + R^2\cos^2 u dv^2\ .$$

(b) The torus with inner radius, r and outer radius, R

(solution) the equation

$$\left.\begin{aligned} x &= (a + b\cos u)\cos v \\ y &= (a + b\cos u)\sin v \\ z &= b\sin u \end{aligned}\right\}$$

Where $a = \frac{R+r}{2},\ b = \frac{R-r}{2}$. Therefor

$$\left.\begin{aligned} x &= \left(\frac{R+r}{2} + \frac{R-r}{2}\cos u\right)\cos v \\ y &= \left(\frac{R+r}{2} + \frac{R-r}{2}\cos u\right)\sin v \\ z &= \frac{R-r}{2}\sin u \end{aligned}\right\}.$$

i.e.

$$\dot{x}_u = \frac{r-R}{2}\sin u\cos v,\ \dot{y}_u = \frac{r-R}{2}\sin u\sin v,\ \dot{z}_u = \frac{R-r}{2}\cos u$$
$$\dot{x}_v = -\left(\frac{R+r}{2} + \frac{R-r}{2}\cos u\right)\sin v,\ \dot{y}_v = \left(\frac{R+r}{2} + \frac{R-r}{2}\cos u\right)\cos v,\ \dot{z}_v = 0$$

$$\therefore g_{11} = \dot{x}_u^2 + \dot{y}_u^2 + \dot{z}_u^2 = \frac{(R-r)^2}{4},\ g_{22} = \frac{1}{4}[(R+r)+(R-r)\cos u],\ g_{12} = 0$$

Therefor the first fundamental form is $\frac{1}{4}\left\{(R-r)^2 du^2 + [(R+r)+(R-r)\cos u]^2 dv^2\right\}$.

2. Use your answers to the previous question to find the length of
   (a) A curve from the north pole of the sphere that winds twice around the sphere before ending up at the south pole
   (b) A curve that winds three times around the small randius for each time around the major radius
3. In lectures we calculated E_1 and shows that for the inertial frame

$$X(x_1,x_2) = \begin{pmatrix} x_1 + r(x_1,x_2) \\ x_2 + s(x_1,x_2) \\ q(x_1,x_2) \end{pmatrix}$$

   then $r_{11}(0,0) = 0$. By calculating E_2, F_1, F_2, G_1, and G_2, show that all the second derivatives of r and s are zero at (0,0).
4. Show that $F_{12} - \frac{1}{2}F_{22} - \frac{1}{2}G_{11} - q_{11}q_{22} - q_{12}^2$.

Tutorial 4

1. Construct an atlas for
   (a) The torus
   (b) the cylinder

   from the charts for the circle from the lectures.

(solution) (a) Let S^1 be the circumference and $M = S^1 \times S^1$.

$U \subset S^1,\ V \subset S^1$

$\varphi : U \to [-1,1] \qquad (U,\varphi)$

$\psi : V \to [-1,1] \qquad (V,\psi)$

$\therefore$ the atlas is $\{(U\times V,\ \varphi\times\psi)\}$.

(b) Let S^1 be the circumference and I be the open interval, $M = S^1\times S^1$.

$$U\subset S^1,\ \ V\subset I$$
$$\varphi : U\to[-1,1]$$
$$\psi : V\to(a,b)$$

$$\therefore\{(U\times V,\ \varphi\times\psi)\}$$

2. Show that the function on the sphere that outputs the $z-$coordinate of the point is differentiable.
(proof) the spherical co-ordinates

$$\left.\begin{aligned} x &= a\cos u\cos v\\ y &= a\cos u\sin v\\ z &= a\sin u\end{aligned}\right\}$$

$\dfrac{dz}{du} = a\cos u$. Therefor the function is differentiable.

3. Show that function on the real projective plane given by the angle the line makes with the $xy-$plane is differentiable.

(proof) $f:(x,y,z)\to \arcsin\dfrac{a}{\sqrt{x^2+y^2+z^2}}\qquad (xyz\neq 0)$

$$\frac{\partial f}{\partial x} = -\frac{xz}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}$$
$$\frac{\partial f}{\partial y} = -\frac{yz}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}$$
$$\frac{\partial f}{\partial z} = -\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}$$

Because $xyz\neq 0$, the function is differentiable.

Tutorial 5

1. Write the coordinate vector-fields for cartesian coordinates on $\mathbb{R}^2$,

$$\frac{\partial}{\partial x}\ \text{and}\ \frac{\partial}{\partial y}$$

in terms of the polar coordinate vector fields

$$\frac{\partial}{\partial r} \text{ and } \frac{\partial}{\partial \theta}$$

(solution) $\left.\begin{matrix} x = x(r,\theta) \\ y = y(r,\theta) \end{matrix}\right\}, \left.\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}\right\}, \quad r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}, \quad \theta = \arctan\frac{y}{x}$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta, \quad \frac{\partial r}{\partial y} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

Therefor

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\cdot\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\cdot\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\cdot\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\cdot\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial}{\partial \theta}$$

2. Calculate the vector-field transformation between stereographic coordinates and the angular coordinates on the sphere, S^2

Tutorial 6

Let $A = y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}, \ B = x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}, \ C = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.

1. Calculate the Lie derivative of B with respect to A.

(solution)

$$L_A B = [A,B] = AB - BA$$
$$= \left(y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right) - \left(x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right)\left(y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}\right)$$
$$= y\frac{\partial}{\partial z}\left(x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right) + z\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right) -$$
$$- x\frac{\partial}{\partial z}\left(y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}\right) - z\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}\right)$$
$$= yx\frac{\partial^2}{\partial z^2} + y\frac{\partial z}{\partial z}\cdot\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial z\partial x} + zx\frac{\partial^2}{\partial y\partial z} + z^2\frac{\partial^2}{\partial y\partial x} -$$
$$- xy\frac{\partial^2}{\partial z^2} - x\frac{\partial z}{\partial z}\cdot\frac{\partial}{\partial y} - xz\frac{\partial^2}{\partial z\partial y} - zy\frac{\partial^2}{\partial x\partial z} - z^2\frac{\partial^2}{\partial x\partial y}$$
$$= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

2. Show $Af = Bf = Cf$ for $f(x,y,z) = x^2 + y^2 - z^2$.

(proof) $f(x,y,z) = x^2 + y^2 - z^2$

$$Af = \left(y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}\right)(x^2 + y^2 - z^2) = y(-2z) + z\cdot 2y = 0$$
$$Bf = \left(x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right)(x^2 + y^2 - z^2) = x(-2z) + z\cdot 2x = 0$$
$$Cf = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)(x^2 + y^2 - z^2) = x\cdot 2y - y\cdot 2x = 0$$

3. Use that fact to sketch the curves of the one parameter groups associated with A, B, and C.

(solution) curve C : $\left.\begin{matrix} x = x(z) \\ y = y(z) \end{matrix}\right\}$ (where z is auxiliary variable)

The one parameter groups are $f_1 : z \to y,\ f_2 : y \to x$.

$y = f_1(z),\ x = f_2(f_1(z)) = f_1 \circ f_2(z)$

$f(x,y,z) = x^2(z) + y^2(z) - z^2$

$$Af = y\frac{\partial f}{\partial z} + z\frac{\partial f}{\partial y} = y(-2z) + z\cdot 2y\cdot\frac{dy}{dz} = 0 \qquad \text{.......................................(1)}$$

$$Bf = x(-2z) + z\cdot 2x\cdot\frac{dx}{dz} = 0 \qquad \text{.......................................(2)}$$

$$Cf = x\cdot 2y\cdot\frac{dy}{dz} - y2x\cdot\frac{dx}{dz} = 0 \qquad \text{.......................................(3)}$$

From (3), $\frac{dy}{dx} = \frac{dx}{dz}$.

(1), (2) $\Rightarrow \frac{dy}{dz} = \frac{dx}{dz} = 1 \Rightarrow dx = dy = dz$

The tangent vector of the curve $\begin{cases} x = \varphi(t) \\ y = \psi(y) \\ z = t \end{cases}$, $\vec{\tau} = (1,1,1)$

$\frac{dy}{dz} = 1 \Rightarrow y = z + c_1$

$\frac{dx}{dz} = 1 \Rightarrow x = z + c_2$

$\therefore \begin{cases} x = t + c_1 \\ y = t + c_2 \\ z = t \end{cases}$

When $c_1 = c_2 = 0$, the curve C is the line parallel to $\vec{\tau}$ and passing (0,0,0).

In general, the curve C is the line parallel to $\vec{\tau}$ and passing $(c_1, c_2, 0)$.

Tutorial 7

Let M be a two-dimensional manifold with coordinates x_1 and x_2. The Christoffel symbols for a connection ∇ are identically zero except for

$$\Gamma^1_{12} = \Gamma^1_{21} = -\tan x_2,\ \Gamma^2_{11} = \cos x_2 \sin x_2$$

1. Calculate $\nabla_X Y$ for

$$X = \frac{\partial}{\partial x_1},\ Y = \frac{1}{\cos x_2}\frac{\partial}{\partial x_1}.$$

(solution) $\nabla_X Y \nabla_i Y^j = x^i \frac{\partial}{\partial x^j}$

$$X^1 = 1,\ X^2 = 0,\ Y^1 = \frac{1}{\cos x_2},\ Y^2 = 0$$

$$\nabla_i Y^j = \frac{\partial Y^j}{\partial x^i} + \Gamma^j_{ik} Y^k$$

$$\nabla_1 Y^1 = \frac{\partial Y^1}{\partial x^1} + \Gamma_{11}^1 Y^1 + \Gamma_{12}^1 Y^2 = 0 + 0 + \tan x_2 \cdot 0 = 0$$

$$\nabla_1 Y^2 = \frac{\partial Y^2}{\partial x^1} + \Gamma_{11}^2 Y^1 + \Gamma_{12}^2 Y^2 = 0 + \cos x_2 \sin x_2 \cdot \frac{1}{\cos x_2} + 0 = \sin x_2$$

$$\nabla_2 Y^1 = \frac{\partial Y^1}{\partial x^2} + \Gamma_{21}^1 Y^1 + \Gamma_{22}^1 Y^2 = \frac{\partial}{\partial x^2}\left(\frac{1}{\cos x_2}\right) + (-\tan x_2) \cdot \frac{1}{\cos x_2} + 0 = 0$$

$$\nabla_2 Y^2 = \frac{\partial Y^2}{\partial x^2} + \Gamma_{21}^2 Y^1 + \Gamma_{22}^2 Y^2 = 0 + 0 + 0 = 0$$

$$\therefore \nabla_X Y = X^i \nabla_i Y^j \frac{\partial}{\partial x^j} = X^1 \nabla_1 Y^2 \frac{\partial}{\partial x^2} = 1 \cdot \sin x_2 \frac{\partial}{\partial x^2} = \sin x_2 \frac{\partial}{\partial x^2}.$$

2. Write down the equations for parallel transport for this connection.

   (solution) for parallel transport, $\nabla_X Y = 0 \Rightarrow \sin x_2 \frac{\partial}{\partial x^2} = 0$

   Assume that the vector field $Y(t)$ parallel transport according to the curve r.

   $$r: \left.\begin{matrix} x_1 = x_1(t) \\ x_2 = x_2(t) \end{matrix}\right\} !$$

   $$\left.\begin{matrix} -\frac{\sin x_2}{\cos^2 x_2} \frac{dx_2}{dt} = 0 \\ \sin x_2 \frac{dx_1}{dt} = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x_2(t) = c_2 \\ x_1(t) = c_1 \end{matrix}\right\}$$

3. Combine them into a single equation and write down the solution.
   (solution)
4. Pick a starting point and vector and solve for the coefficients in the solution.
5. Calculate the torsion of this connection.

   (solution) $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$: torsion tensor

   $\Gamma_{ij}^k = 0$. $\therefore$ torsion=0

Tutorial 8

1. Write the standard metric for the sphere in terms of the coordinates θ and ϕ.

   (solution)

   $$\left.\begin{matrix} x = \cos\varphi \cos\theta \\ y = \cos\varphi \sin\theta \\ z = \sin\varphi \end{matrix}\right\}$$

   the standard metric:

   $$\overrightarrow{r_\varphi} = (-\sin\varphi\cos\theta, -\sin\varphi\sin\theta, \cos\varphi)$$

   $$\overrightarrow{r_\theta} = (-\cos\varphi\sin\theta, \cos\varphi\cos\theta, 0)$$

   $$g_{11} = 1,\ g_{12} = 0,\ g_{22} = \cos^2\varphi,\ dS^2 = d\varphi^2 + \cos^2\varphi d\theta^2$$

2. Write the standard metric for the torus in terms of the toroidal and poloidal angles.

(solution)

$$\left.\begin{array}{l} x=(a+b\cos u)\cos v \\ y=(a+b\cos u)\sin v \\ z=b\sin u \end{array}\right\}$$

$$\vec{r}_u=(\sinh\frac{u}{a}\cos u,\sinh\frac{u}{a}\sin v,1)$$

$$\vec{r}_v=(-a\cosh\frac{u}{a}\sin u,a\cosh\frac{u}{a}\cos v,0)$$

$$g_{11}=\sinh^2\frac{u}{a}+1,\ g_{12}=0,\ g_{22}=a^2\cosh^2\frac{u}{a}$$

$$dS^2=(\sinh^2\frac{u}{a}+1)du^2+(a^2\cosh^2\frac{u}{a})dv^2$$

3. Consider the metric $g=dw^3-dt^2-dz^2$ and the coordinate transformations

$$z(x,y)=\cosh(\varepsilon x)\cos(\varepsilon y)$$
$$t(x,y)=\cosh(\varepsilon x)\sin(\varepsilon y)$$
$$w(x,y)=\sinh(\varepsilon x)$$

(a) Calculate $z^2+t^2+w^2$

(b) Express g in the new coordinates

(solution)

$$\left.\begin{array}{l} z'_x=\varepsilon\sinh(\varepsilon x)\cos(\varepsilon y) \\ t'_x=\varepsilon\sinh(\varepsilon x)\sin(\varepsilon y) \\ w'_x=\varepsilon\cosh(\varepsilon x) \end{array}\right\},\ \left.\begin{array}{l} z'_y=-\varepsilon\cosh(\varepsilon x)\sin(\varepsilon y) \\ t'_y=\varepsilon\cosh(\varepsilon x)\cos(\varepsilon y) \\ w'_y=0 \end{array}\right\}$$

(a) $\therefore z^2+t^2+w^2=\cosh(2\varepsilon x)$

(b)

$$\left.\begin{array}{l} dz=z'_x dx+z'_y dy \\ dw=w'_x dx+w'_y dy \\ dt=t'_x dx+t'_y dy \end{array}\right\}$$

$$g=dw^2-dt^2-dz^2=$$
$$=w_x'^2dx^2+2w'_xw'_ydxdy+w_y'^2dy^2-(t_x'^2dx^2+2t'_xt'_ydxdy+t_y'^2dy^2)$$
$$-(z_x'^2dx^2+2z'_xz'_ydxdy+z_y'^2dy^2)=$$
$$=(w_x'^2-t_x'^2-z_x'^2)dx^2+2(w'_xw'_y-t'_xt'_y-z'_xz'_y)dxdy+(w_y'^2-t_y'^2-z_y'^2)dy^2=$$
$$=\varepsilon^2(\cosh^2(\varepsilon x)-\sinh^2(\varepsilon x))dx^2-\varepsilon^2\cosh^2(\varepsilon x)dy^2=$$
$$=\varepsilon^2\left\{[\cosh^2(\varepsilon x)-\sinh^2(\varepsilon x)]dx^2-\cosh^2(\varepsilon x)dy^2\right\}$$

4. Express the metric for Minkowski space $g=cdt_0^2-dx_0^2-dy_0^2-dz_0^2$ in terms of new coordinates

$t_0 = t$

$x_0 = r\cos(\phi + \omega t)$

$y_0 = r\sin(\phi + \omega t)$

$z = z$

(solution) $g = cdt_0^2 - dx_0^2 - dy_0^2 - dz_0^2$

$$\begin{cases} t'_{0z} = 0 \\ x'_{0z} = 0 \\ y'_{0z} = 0 \\ z'_{0z} = 1 \end{cases}, \begin{cases} t'_{0\phi} = 0 \\ x'_{0\phi} = -r\sin(\phi + \omega t) \\ y'_{0\phi} = r\cos(\phi + \omega t) \\ z'_{0\phi} = 0 \end{cases}, \begin{cases} t'_{0t} = 1 \\ x'_{0t} = -\omega r\sin(\phi + \omega t) \\ y'_{0t} = \omega r\cos(\phi + \omega t) \\ z'_{0t} = 0 \end{cases}$$

$dt = t'_{0\phi}d\phi + t'_{0t}dt + t'_{0z}dz = dt$

$dx_0 = x'_{0\phi}d\phi + x'_{0t}dt + x'_{0z}dz = -r\sin(\phi + \omega t)d\phi - \omega r\sin(\phi + \omega t)dt = -r\sin(\phi + \omega t)(d\phi + \omega dt)$

$dy_0 = y'_{0\phi}d\phi + y'_{0t}dt + y'_{0z}dz = r\cos(\phi + \omega t)d\phi + \omega r\cos(\phi + \omega t)dt = r\cos(\phi + \omega t)(d\phi + \omega dt)$

$dz_0 = z'_{0\phi}d\phi + z'_{0t}dt + z'_{0z}dz = dz$

$$\begin{aligned} g &= cdt_0^2 - dx_0^2 - dy_0^2 - dz_0^2 = \\ &= cdt^2 - r^2\sin^2(\phi + \omega t)(d\phi^2 + 2\omega d\phi dt + \omega^2 dt^2) - r^2\cos^2(\phi + \omega t)(d\phi^2 + 2\omega d\phi dt + \omega^2 dt^2) - dz^2 \\ &= cdt^2 - r^2(d\phi^2 + 2\omega d\phi dt + \omega^2 dt^2) - dz^2 \\ &= (c - \omega^2 r^2)dt^2 - r^2 d\phi^2 - 2\omega r^2 d\phi dt - dz^2 \end{aligned}$$