Topology Assignment

October 2017

Problem 1

1(a)

Yes, the space (X, T) is compact, because the set is closed and bounded and is a subset of the Euclidean space.

1(b)

The identity mapping from the set (X, d) to (X, ρ) is continuous. Hence, any subset of (X, d) that is compact in the former space is also compact in the latter space. Thus, the space (X, T) is compact under the new metric.

1(c)

By discrete nature of the metric, the only open sets are singleton sets. Thus, an open cover of the space would be $\{\{x\} : x \in X\}$. The open cover does not have a finite sub-cover, as the given space is infinite.

1(d)

The space (X, T) is compact. This is because every open cover should contain X, which can cover the space by itself singly, thus forming a finite subcover.

1(e)

The set X is not compact. Consider the open cover $\{\phi\} \cup \{\{\frac{1}{2017}, x\} : x \neq \frac{1}{2017}, x \in X\}$. This has no finite subcover.

1(f)

This case is similar to part $1(a)$ above and hence is compact.

1(g)

The space (X, T) is compact, because every open cover consist of elements that miss finitely many points of the set X. Choosing one of the element and other finite elements in the cover that have at least exactly those points would give us a finite cover, thus proving compactness.

Problem 2

2(a)

Let $x, y \in X$. Fix an ϵ. Then, $x \in B_\epsilon(x_1)$ and $y \in B_\epsilon(x_2)$ where $x_1, x_2 \in F$. Now, by triangle inequality,

$$d(x, y) \leq d(x, x_1) + d(x_1, x_2) + d(x_2, y)$$

$$\implies d(x, y) \leq \epsilon + k + \epsilon = M$$

, where k is a finite number by virtue of the finiteness of the set F. Thus, boundedness is proved.

2(b)

As shown in the hint, let (X, d) be not totally bounded, i.e., $\exists \epsilon : X$ cannot be covered by $B_\epsilon(\xi), \xi \in F$ where F is some finite set. Now, choose the sequence $\{x_n\}$ as follows: $x_1 \in X, x_2 \in X/B_\epsilon(x_1), x_3 \in X/B_\epsilon(x_1)/B_\epsilon(x_2)$ and so on.Then, we have $d(x_n, x_m) \geq \epsilon$ for any pair of integers since x cannot be covered by the finite sequence of balls. Hence, X has a sequence with no limit point, which is a contradiction.

Problem 3

3(a)

Let $x \in K$ and $\xi \in X/K$. Now, by the Hausdorff condition, we can choose disjoint neighbourhoods for each of the points $x \in K$ and $\xi \in X/K$, U_x and U_y of x and ξ respectively. Now, $\{U_x : x \in K\}$ is an open cover of K and has admits of a finite cover by the compactness condition, say $\{U_{x_i} : i = 1, 2, \ldots, n\}$. Then, $U = \cup_{x_i} U_{x_i}$ is an open set such that $K \subset U$ and it is disjoint from the open set $V = \cap_{y_i} U_{y_i}$ formed by taking intersections with each corresponding disjoint neighbourhood U_{x_i} respectively. For, if $z \in U$, then $z \notin U_{x_i}$ for some i, hence $z \notin U_{y_i}$ thus $z \notin V$. Hence, U, V are the desired open subsets.

3(b)

By the above problem, for every $\xi \in X/K$, there is a disjoint neighbourhood or open set disjoint from K, thus proving that X/K is open set, hence K is closed.

Problem 4

By the preceding problem, since $C \in X/K$, therefore there exist open sets U, V such that $C \subset U$ and $K \subset V$ and U, V are disjoint.