Mathematics – Numerical Analysis Solution

Problem 1:

$$\begin{cases} 0 \le x \le a \\ 0 \le y \le b \\ U(x,0) = 0 \\ U(a,y) = 0 \\ U(x,b) = \dfrac{U_0 x(a-x)}{a^2} \\ U(0,y) = 0 \end{cases}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad [1]$$

The solution of Laplace's equation proceeds by a method known as the separation of variables. In this method we postulate a solution that is the product of two functions, h(x) a function of x only and ф(y) a function of the y only. With this assumption, our solution becomes.

$U(x,y) = h(x)\phi(y)$ *[2]*

We do not know, in advance, if this solution will work. However, we assume that it will and we substitute it for u in equation [1]. Since X(x) is a function of x only and Y(y) is a function of y only, we obtain the following result when we substitute equation [2] into equation [1].

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \left[h(x)\phi(y)\right]}{\partial x^2} + \frac{\partial^2 \left[h(x)\phi(y)\right]}{\partial y^2} = \phi(y)\frac{\partial^2 h(x)}{\partial x^2} + h(x)\frac{\partial^2 \phi(y)}{\partial y^2} = 0 \quad [3]$$

$$\frac{1}{h(x)}\frac{\partial^2 h(x)}{\partial x^2} = -\frac{1}{\phi(y)}\frac{\partial^2 \phi(y)}{\partial y^2} \quad [4]$$

The left hand side of equation [4] is a function of x only; the right hand side is a function of y only. The only way that this can be correct is if both sides equal a constant. This also shows that the separation of variables solution works. In order to simply the solution, we choose the constant[1] to be equal to μ. This gives us two ordinary differential equations to solve.

$$\frac{h''(x)}{h(x)} = -\frac{\phi''(y)}{\phi(y)} = \mu$$

The separation yields the following problem for h and Φ:

$$h''(x) - h(x)\mu = 0, h(0) = h(a) = 0$$

And $\phi''(y) + \phi(y)\mu = 0, \phi(0) = 0$

$$\frac{d^2h(x)}{dx^2} + \mu h(x) = 0 \ \rightarrow \ h(x) = Asin(\sqrt{\mu}x) + Bcos(\sqrt{\mu}x) \quad [6]$$

$$\frac{d^2\phi(y)}{dx^2} + \mu\phi(y) = 0 \ \rightarrow \ \phi(y) = Csinh(\sqrt{\mu}y) + Dcosh(\sqrt{\mu}y) \quad [7]$$

From the solutions in equations [6] and [7], we can write the general solution for u(x,y) = h(x)ϕ(y) as follows.

$$U(x,y) = (Asin(\sqrt{\mu}x) + Bcos(\sqrt{\mu}x))(Csinh(\sqrt{\mu}y) + Dcosh(\sqrt{\mu}y)) \quad [8]$$

We now apply the boundary conditions shown with the original equation [1] to evaluate the constants A, B, C, and D.

$$U(0,y) = (Asin(\sqrt{\mu}0) + Bcos(\sqrt{\mu}0))(Csinh(\sqrt{\mu}y) + Dcosh(\sqrt{\mu}y)) \quad [9]$$

Because sin(0) = 0 and cos(0) = 1, equation [9] will be satisfied for all y only if B = 0. Thus, we set B = 0. Next we apply the solution in equation [8] (with B = 0) to the boundary condition at y = 0

$$U(x,0) = (Asin(\sqrt{\mu}x))(Csinh(\sqrt{\mu}0) + Dcosh(\sqrt{\mu}0)) \quad [10]$$

Since sinh(0) = 0 and cosh(0) = 1, this boundary condition will be satisfied only if D = 0. The third boundary condition that u = 0 occurs at x = a. At this point we have the following result, using solution in [8] with B = 0 and D = 0 as found previously.

$$U(a,y) = (Asin(\sqrt{\mu}x))(Csinh(\sqrt{\mu}y)) \quad [11]$$

General solution will look:

U(x, y)=$\sum_{n=1}^{\infty} a_n \sin(\frac{\pi nx}{a}) sinh(\frac{\pi ny}{a})$

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[1] The choice of μ for the constant as opposed to just plain λ comes from experience. Choosing the constant to have this form now gives a more convenient result later. In solutions to Laplace's equation in rectangular geometries we will select the constant as μ to give us an ordinary differential equation whose solutions results in sines and cosines in the direction for which we have a Sturm-Liouville problem (with homogenous boundary conditions).

$U(x, b) = \sum_{n=1}^{\infty} a_n \sin(\frac{\pi nx}{a}) sinh\left(\frac{\pi nb}{a}\right) = \frac{U_0 x(a-x)}{a^2}$

$$a_n\left(\frac{a}{2}\right) sinh\left(\frac{\pi nb}{a}\right) = \int_0^a \frac{U_0 x(a-x)}{a^2} sin\left(\frac{\pi nx}{a}\right) dx$$

$$a_n = \frac{2}{a sinh\left(\frac{\pi nb}{a}\right)} \int_0^a \frac{U_0 x(a-x)}{a^2} sin\left(\frac{\pi nx}{a}\right) dx$$

The same approach (as in the problem 1) is used in all following problems!

Problem 2:

$$\begin{cases} 0 \le x \le a \\ 0 \le y \le b \\ U(x,0) = 0 \\ \frac{\partial u}{\partial x}(a,y) = q_0 \\ U(x,b) = 0 \\ U(0,y) = U_0\left(0,\frac{y}{b}\right)(1-\frac{y}{b}) \end{cases}$$

The solution we will find in this way:

$U(x,y) = h(y)\phi(x)$

$\frac{h''(x)}{h(x)} = -\frac{\phi''(y)}{\phi(y)} = \mu$

The separation yields the following problem for h and Φ:

$h''(x) + h(x)\mu = 0, h(0) = h(b) = 0$

And $\phi''(y) - \phi(y)\mu = 0, \phi(0) = 0$

General solution will look:

$U(x, y) = \sum_{n=1}^{\infty} a_n \sin(\frac{\pi ny}{b}) \frac{sinh(\frac{\pi n(a-x)}{b})}{sinh(\frac{\pi na}{b})} + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi ny}{b}) \frac{sinh(\frac{\pi nx}{b})}{sinh(\frac{\pi na}{b})}$

$\frac{\partial u}{\partial x}(a, y) = q_0 = \sum_{n=1}^{\infty} b_n \frac{\pi n}{b} \cos(\frac{\pi ny}{b}) \frac{sinh(\frac{\pi na}{b})}{sinh(\frac{\pi na}{b})} = \sum_{n=1}^{\infty} b_n \frac{\pi n}{b} \cos(\frac{\pi ny}{b})$

$$U_0\left(0,\frac{y}{b}\right)\left(1-\frac{y}{b}\right) = \sum_{n=1}^{\infty} a_n \sin(\frac{\pi ny}{b})$$

$$a_n = \frac{2}{a}\int_0^a U_0\left(0, \frac{y}{b}\right)\left(1 - \frac{y}{b}\right) sin\left(\frac{\pi n y}{b}\right) dy$$

$$b_n = \frac{1}{b}\int_0^b q_0 dx$$

Problem 3:

$$\begin{cases} 0 \le x \le a \\ 0 \le y \le b \\ \frac{\partial u}{\partial y}(x, 0) = -q_0 \\ U(a, y) = U_0(a, y) \\ U(x, b) = 0 \\ U(0, y) = 0 \end{cases}$$

The solution we will find in this way:

$U(x, y) = h(y)\phi(x)$

$\frac{h''(x)}{h(x)} = -\frac{\phi''(y)}{\phi(y)} = \mu$

The separation yields the following problem for h and Φ:

$h''(x) + h(x)\mu = 0, h(0) = h(b) = 0$

And $\phi''(y) - \phi(y)\mu = 0, \phi(0) = 0$

General solution will look:

U(x, y)= $\sum_{n=1}^{\infty} a_n \sin(\frac{\pi n y}{b}) \frac{sinh(\frac{\pi n(a-x)}{b})}{sinh(\frac{\pi n a}{b})} + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi n y}{b}) \frac{sinh(\frac{\pi n x}{b})}{sinh(\frac{\pi n a}{b})}$

U_0 *(a, y)*= $q_0 = \sum_{n=1}^{\infty} b_n \sin(\frac{\pi n y}{b})$

$-q_0 = \frac{\partial u}{\partial y}(x, 0) = \sum_{n=1}^{\infty} b_n \frac{\pi n}{b} \frac{sinh(\frac{\pi n x}{b})}{sinh(\frac{\pi n a}{b})}$

$$a_n = \frac{2}{b}\int_0^b U_0(a, y) sin\left(\frac{\pi n y}{b}\right) dy$$

$$b_n = \frac{2}{a}\int_0^a -q_0 sin\left(\frac{\pi n y}{b}\right) dy$$

Problem 4:

$$\begin{cases} 0 \le x \le a \\ 0 \le y \le b \\ U(x,0) = 0 \\ \frac{\partial u}{\partial x}(a,y) = q_0 \\ \frac{\partial u}{\partial y}(x,b) = q_0 \\ U(0,y) = 0 \end{cases}$$

The solution we will find in this way:

$U(x,y) = h(y)\phi(x)$

$\frac{h''(x)}{h(x)} = -\frac{\phi''(y)}{\phi(y)}$=$\mu$

The separation yields the following problem for h and Φ:

$h''(x) + h(x)\mu = 0, h(0) = h(b) = 0$

And $\phi''(y) - \phi(y)\mu = 0, \phi(0) = 0$

General solution will look:

U(x, y)=$\sum_{n=1}^{\infty} a_n \sin(\frac{\pi n y}{b}) \frac{sinh(\frac{\pi n(a-x)}{b})}{sinh(\frac{\pi n a}{b})} + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi n y}{b}) \frac{sinh(\frac{\pi n x}{b})}{sinh(\frac{\pi n a}{b})}$

U_0 *(a, y)*=q_0=$\sum_{n=1}^{\infty} -a_n\, sin\left(\frac{\pi n y}{b}\right) \frac{1}{sinh(\frac{\pi n b}{a})} + \sum_{n=1}^{\infty} \sin(\frac{\pi n y}{b}) \frac{\frac{\pi n}{b} cosh(\frac{\pi n a}{b})}{sinh(\frac{\pi n a}{b})}$

$q_0 = \frac{\partial u}{\partial y}(x,b)$= $\sum_{n=1}^{\infty} a_n \frac{\pi n}{b} cos\left(\frac{\pi n b}{b}\right) \frac{\sinh\left(\frac{\pi n(x-a)}{b}\right)}{\sinh\left(\frac{\pi n b}{a}\right)} + \sum_{n=1}^{\infty} b_n \frac{\pi n}{b} \cos(\frac{\pi n b}{b}) \frac{sinh(\frac{\pi n(a-x)}{b})}{sinh(\frac{\pi n b}{a})}$

$$a_n = \frac{2}{b}\int_0^b q_0 sin\left(\frac{\pi n y}{b}\right) dy$$

$$b_n = \frac{2}{a}\int_0^a q_0 sin\left(\frac{\pi n x}{a}\right) dx$$

Problem 5:

$$\begin{cases} 0 \le x \le a \\ 0 \le y \le b \\ \frac{\partial u}{\partial y}(x,0) = 0 \\ \frac{k}{h}\frac{\partial u}{\partial x}(a,y) + U(a,y) = U_0 \\ U(x,b) = 0 \\ \frac{\partial U}{\partial x}(0,y) = 0 \end{cases}$$

The solution we will find in this way:

$U(x,y) = h(y)\phi(x)$

$\frac{h''(x)}{h(x)} = -\frac{\phi''(y)}{\phi(y)}$=$\mu$

The separation yields the following problem for h and Φ:

$h''(x) + h(x)\mu = 0, h(0) = h(b) = 0$

And $\phi''(y) - \phi(y)\mu = 0, \phi(0) = 0$

General solution will look:

U(x, y)=$\sum_{n=1}^{\infty} a_n \sin(\frac{\pi n y}{b}) \frac{sinh(\frac{\pi n(a-x)}{b})}{sinh(\frac{\pi n a}{b})} + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi n y}{b}) \frac{sinh(\frac{\pi n x}{b})}{sinh(\frac{\pi n a}{b})}$

U_0 =q_0=$\sum_{n=1}^{\infty} b_n\, sin\left(\frac{\pi n y}{b}\right) + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi n y}{b}) \frac{\frac{\pi n}{b} cosh(\frac{\pi n a}{b})}{sinh(\frac{\pi n a}{b})}$

$$b_n = \frac{2}{b}\int_0^b U_0 sin\left(\frac{\pi n y}{a}\right) dy$$

Problem 6:

$$\begin{cases} 0 \le x \le \infty \\ 0 \le y \le b \\ U(x,0) = 0 \\ U(a,y) \to 0 \\ U(x,b) = U_0(x,b)\exp(-cx) \\ U(0,y) = U_0(0,y)\dfrac{y}{b} \end{cases}$$

The solution we will find in this way:

$U(x,y) = h(y)\phi(x)$

$\frac{h''(x)}{h(x)} = -\frac{\phi''(y)}{\phi(y)} = \mu$

The separation yields the following problem for h and Φ:

$h''(x) + h(x)\mu = 0, h(0) = h(b) = 0$

And $\phi''(y) - \phi(y)\mu = 0, \phi(0) = 0$

General solution will look:

U(x, y)= $\sum_{n=1}^{\infty} b_n\, exp\left(-\frac{\pi n x}{b}\right) sin\left(\frac{\pi n y}{b}\right)$

$$b_n = \frac{2}{b}\int_0^b U_0(0,y)\frac{y}{b} sin\left(\frac{\pi n y}{b}\right) dy$$

Problem 7:

The solution we will find in the next way:

$T(x,t) = \varphi(x)G(t)$

$\frac{dG}{dt} = -k\alpha G \ and \ \frac{d^2\varphi}{dx^2} + \alpha\varphi = 0$

List of eigenvalues:

$\alpha_n = \pm\frac{arccos^2\frac{T_0}{c_1}}{l^2}, \alpha_0 = 0$

$T(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\arccos\left(\frac{T_0}{c_1}\right)\frac{x}{l}\right)\exp(-k\left(\frac{arccos^2\frac{T_0}{c_1}}{l^2}\right)t)$

$T(x,0) = \sum_{n=0}^{\infty} A_n \cos\left(\arccos\left(\frac{T_0}{c_1}\right)\frac{x}{l}\right)$

$$A_n = \begin{cases} \frac{1}{l}\int_0^l U_0\left(\frac{x}{l}\right)dx \\ \frac{2}{l}\int_0^l U_0\left(\frac{x}{l}\right)dx\cos\left(\arccos\left(\frac{T_0}{c_1}\right)\frac{x}{l}\right), n \neq 0 \end{cases}$$

Problem 8:

The definite solution of the Poisson's equation could be find in this way (this is a generalized case of solution of Poisson's equation these boundaries):

$$\varphi(x,y) = \sum_{i=0,j=0}^{\infty} \frac{16}{\pi^2(2i+a)(2j+b)((2i+a)^2+(2j+b)^2)} sin\pi(2i+a)xsin(2j+b)y$$