# Assignment 3rd-Aug

August 2017

## 1 Theorem 3.1

### 1.1 Statement

Every path $P_n$ is an integer sum graph.

### 1.2 Proof

A path is defined as a finite or infinite sequence of edges which connect distinct vertices. From the paper[1], it is clear that the set of vertices $\{0\}, \{0, 1\}, \{1, 0, 2\}, \{1, 2, -1, 3\}$ describe the integer sum labelling of the vertices for paths $P_n$ with $n = 1, 2, 3, 4$ because the vertices are labeled distinctly and the sum of any two consecutive terms lie in the set, which is seen by adding two consecutive integers in the above sets, i.e., $0 = 0$;$0 + 1 = 1$;$(-1) + 0 = -1$,$0 + 2 = 2$ and $1 + 2 = 3, 2 + (-1) = 1, 3 + (-1) = 2$ respectively and every edge has, as its vertices labelled such that their sum is in the set.

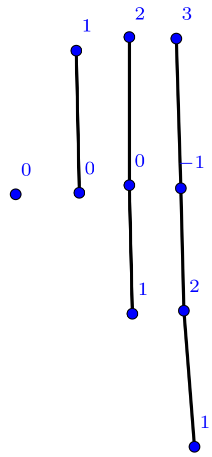


**Fig.1 Different paths with integral sum labelling $P_n$ for $n = 1, 2, 3, 4$**

For extending the above sequence of vertices for $n \geq 4$, as observed in the paper[1], we extend the sequence $\{1, 2, -1, 3, -4, 7, -11, 18, -29, \cdots\}$ where the first two terms are $1, 2$ and the remaining terms are given by the sequence $\{-1, 3, -4, 7, -11, \cdots\}$ with the sequence defined by $a_n = a_{n-2} - a_{n-1},\ n \geq 3$ and $a_1 = -1, a_2 = 3$. The sequence described is an alternating sequence with the absolute values of the terms strictly being increasing, thus being distinct. In addition, the sum of any two consecutive terms is $a_n + a_{n-1} = a_{n-2} - a_{n-1} + a_{n-1} = a_{n-2}$ which clearly lies in the set. In addition, by using the method of difference equations and recursions, it would be observed that sum of no non-adjacent vertices lie in the set. Hence, the extension of labeling is seen to satisfy the condition for an integer sum graph.

# 2 Theorem 3.2

## 2.1 Statement

Every matching is an integer sum graph

## 2.2 Proof

An $n$-matching is a collection of $n$ independent or non-adjacent lines which are represented by sequences of pairs of distinct integers. Again, from the referred paper[1], it is clear that the set of pairs of sequences $\{\{0, 1\}, \{\{-1, 2\}, \{-2, 1\}\}, \{\{-1, 2\}, \{-2, 1\}, \{7, -5\}\}$

define an integer sum labeling of the $n$-matchings for $n = 1, 2, 3$. This is because the sum of numbers in any one pair again lies in one of the pairs of the set. This is seen by noting that $0+1 = 1$;$(-1)+2 = 1, (-2)+1 = -1$;$(-1)+2 = 1, (-2) + 1 = 1, 7 + (-5) = 2$ respectively and there exist no other edges whose sum of vertices does not belong to the set of pairs. Now, to extend the above sequence for $n \geq 3$, we construct a sequence of pairs as follows:

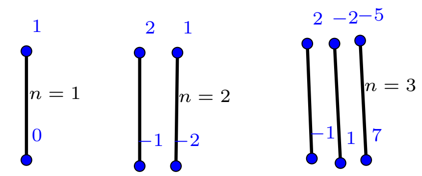


**Fig.2 integer sum labeling of vertices for matchings in cases** $n = 1, 2, 3$

$\{\{-1, 2\}, \{1, -2\}, \{7, -5\}, \{49, -42\}, \{343, -294\} \cdots\}$ where the first three terms are {-1,2},{1,-2},{7,-5} and the first number of the next terms are determined by multiplying by 7 to the first number of the previous pair, and the second number is determined as the negative value of the number obtained by substracting the hitherto obtained number from first number of the previous pair.Thus, for example, the fourth pair is obtained by $7 \times 7 = 49$(first number) and $-(7 \times 7 - 7) = -42$(second number). Similarly, fifth pair is obtained as $7 \times 49 = 343$(first number) and $-(343 - 49) = -294$(second number) By doing so, the absolute values of the numbers in each pairs would be an increasing sequence thereby ensuring distinctness and since the sum of two numbers of any pair is equal to the first number of the previous pair,and again, by using recursion and properties of integers, it can be seen that no other sum lies in any pair. Thus, the condition for integer sum labeling is satisfied. Thus, every $n$-matching can be made into an integer sum graph.

# 3 On Conjecture 5.1

## 3.1 Statement

Every tree with $\zeta(T) = 0$ is a caterpillar.

## 3.2 Disproof

It is proved in Chen's paper[2] that there are infinitely many trees $T$ which are not caterpillars but are integer sum graphs, i.e. have $\zeta(T) = 0$, thus clearly refuting the conjecture. We closely follow its proof. The integer sum number,$\zeta(G)$ of a graph $G$ is the minimal number(integer) required to make the graph $G \cup \zeta(G)K$ an integer sum graph, where $\zeta(G)K$ refers to independent unconnected nodes of the graph. For integer sum graphs, since no additional nodes are required to make it an integer sum graph, $\zeta(G) = 0$. Now, an $n$-tree is nothing but a connected acyclic graph of $n$ vertices, i.e.. It contains no independent or isolated nodes and does does not have any closed paths or cycles.

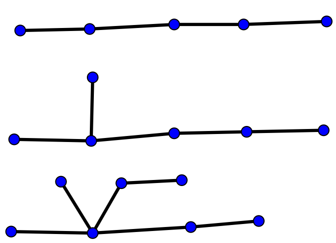

**Fig. 3 Some Common trees**

In addition, when the endpoints of a tree are removed, if it turns to be a path or a linear graph, then the tree is said to be a caterpillar. For example, the figures below represent caterpillars on five vertices.

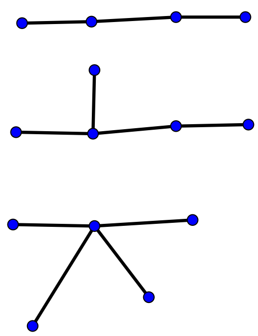

**Fig.4 Caterpillars for five vertices**

Now, let us consider, for $n \geq 3$, the sequence $a_1 = 1, a_2 = 2, a_n = a_{n-1} + a_{n-2}$. Clearly, all terms are distinct as the terms are increasing. Also, as in the theorems above, using difference equations, sum of no non-adjacent vertices lies in the set sequence. In addition, consider the $n$-tree $T$ with vertices as $\{f(a_i)\}$ and edges given by $\{f(a_1)f(a_2), f(a_2)f(a_3)(a_{n-3})f(a_{n-2})\} \cup \{f(a_3)f(a_{n-1}), f(a_{n-1})f(a_n)\}$ for $n \geq 9$, where

$$f(a_i) = \begin{cases} a_i, \ 1 \leq i \leq n-3 \\ 3 - a_{n-3}, \ i = n-2 \\ -a_{n-3}, \ i = n-1 \\ a_{n-2}, \ i = n \end{cases}$$

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The verification that $f$ defines an integer sum labeling for the vertices is straightforward. For, upto vertices $a_{n-3}$, the value of the sum of any two numbers in the labeling is $f(a_i) + f(a_{i-1}) = a_{i+1}$ and lie in the sequence set. For vertices $n-2, n-1$ and $n$, we have to consider the edges $\{a_{n-3}a_{n-2}\}, \{3a_{n-1}\}$ and $\{a_{n-1}a_n$. The sums required, therefore, are $f(a_{n-3}) + f(a_{n-2}) = a_{n-3} + 3 - a_{n-3} = 3, f(3) + f(a_{n-1}) = 3 - a_{n-3}$, and $f(a_{n-1}) + f(a_n) = a_{n-2} - a_{n-3}$, all of which lie in the set of labeling $\{f(a_i)\}$. Also, the sum of $f(a_i), \ i = n-2, n-1, n$ with any other $i$ in the tree node sequence would not lie in the set. Thus, the graph(tree) is an integer sum graph and hence, its integer sum number, $\zeta(T) = 0$. But, it is not a caterpillar, because, by removing the endpoints $a_1, a_{n-2}$ and $a_n$, we do not get a single path. A sample of size $n = 9$ is shown below.

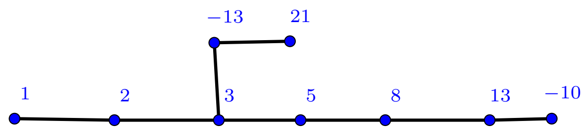


**Fig.5 A tree which is not a caterpillar but an integer sum graph as discussed in the disproof**

# 4 References

[**1**]Harary, Frank. "Sum graphs over all the integers." Discrete Mathematics 124.1-3 (1994): 99-105.

[**2**]Chen, Zhibo. "Harary's conjectures on integral sum graphs." Discrete Mathematics 160.1-3 (1996): 241-244.